Meriam 4/4, 4/5 : System of Particles: Angular Momentum

(a) Fixed Reference Point "O"

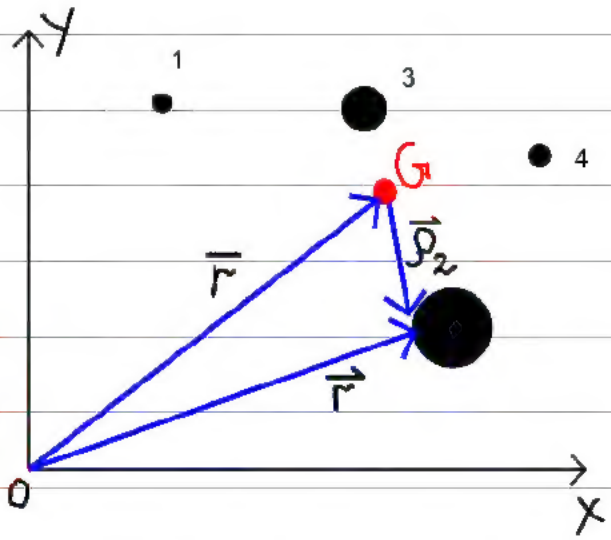
Consider a system of N particles

$\vec{r}_G = \bar{r}$ = position of center of mass (center of gravity)

Let $\vec{\rho}_i$ = position of particle i wrt G

$$\rightarrow \quad \vec{r}_i = \vec{r}_G + \vec{\rho}_i$$

$$\rightarrow \quad \vec{v}_i = \vec{v}_G + \dot{\vec{\rho}}_i$$



$$\boxed{\vec{H}_o = \sum_{i=1}^{N} (\vec{H}_o)_i = \sum_{i=1}^{N} (\vec{r}_i \times m_i \vec{v}_i)}$$

$$\dot{\vec{H}}_o = \frac{d}{dt}\left[\sum_{i=1}^{N} (\vec{r}_i \times m_i \vec{v}_i)\right] = \sum_{i=1}^{N} (\dot{\vec{r}}_i \times m_i \vec{v}_i) + \sum_{i=1}^{N} (\vec{r}_i \times m_i \dot{\vec{v}}_i)$$

($\vec{v}_i$ = $\dot{\vec{r}}_i$; $\dot{\vec{v}}_i$ = $\vec{a}_i$)

$$\boxed{\dot{\vec{H}}_o = \sum_{i=1}^{N} (\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{N} \vec{M}_o}$$

$$\boxed{(\vec{H}_o)_2 = (\vec{H}_o)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (\vec{M}_o)_i \, dt}$$

(b) Reference point G

The equations for $\vec{H}_o$ also work with the center of gravity G as the reference point:

i.e. $$\dot{\vec{H}}_G = \sum_{i=1}^{N} \vec{M}_G \quad \text{, where} \quad \vec{H}_G = \sum_{i=1}^{N} (\vec{\rho}_i \times m_i \vec{V}_i) \quad \bigstar$$

(proof on pages 124-125 of Meriam)

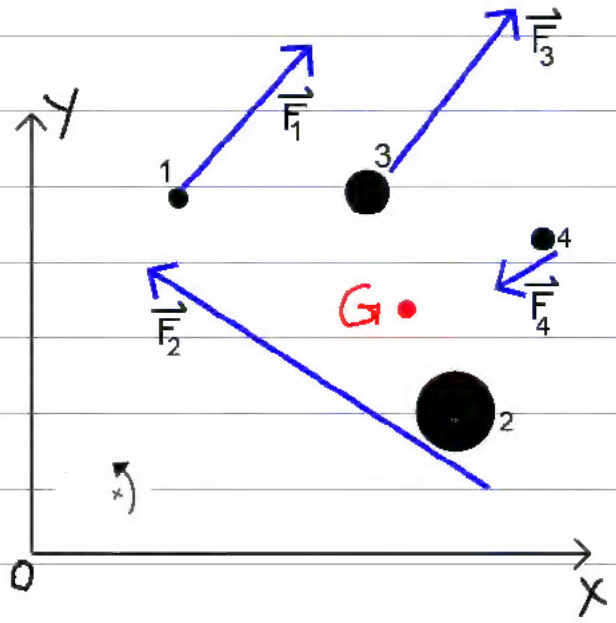
integration of ★ yields:

$$(\vec{H}_G)_2 = (\vec{H}_G)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (\vec{M}_G)_i \, dt$$

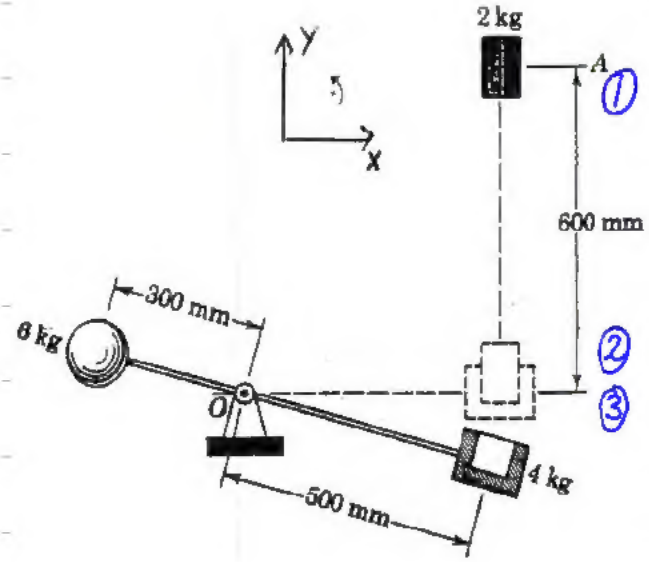
eg.,

$$\sum_{i=1}^{N} \vec{M}_G < 0$$

$$\sum_{i=1}^{N} \vec{M}_o > 0$$



3/234 The 6-kg sphere and 4-kg block (shown in Section) are secured to the arm of negligible mass that rotates in the vertical plane about a horizontal axis at O. The 2-kg plug is released from rest at A and falls into the recess in the block when the arm has reached the horizontal position. An instant before engagement, the arm has an angular velocity: 2 rad/s.



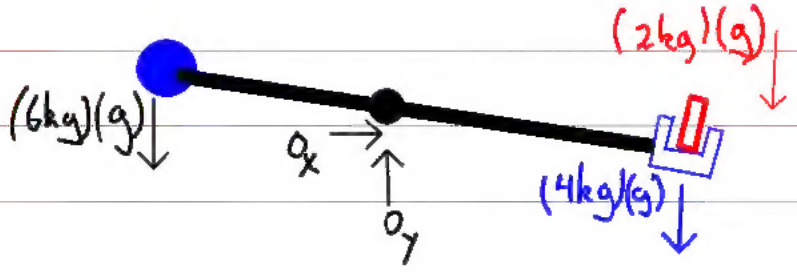
② just before impact
③ just after impact

(a) At what speed v_2 does the 2-kg plug hit the 4-kg block?

constant acceleration → $v_2^2 = v_1^2 + 2a_0\Delta s = (2)(-9.81)(-0.6 \text{ meters})$

→ At time of impact, $\vec{V}_2$ of 2-kg plug is -3.43 m/s $\hat{j}$

(b) Find $\dot{\theta}_3$ (just after impact)

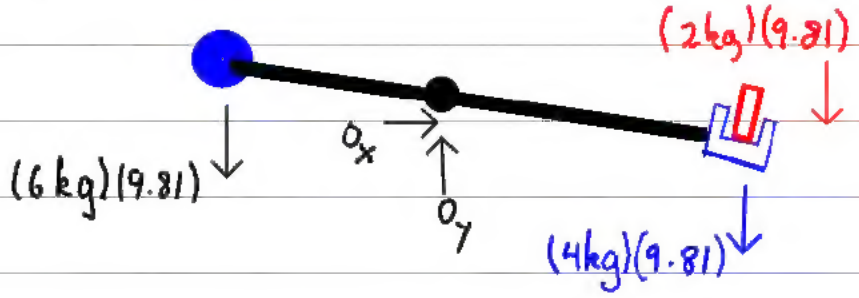


Note:

- Kinetic energy is not conserved during impact
- $\vec{G}_3 = \vec{G}_2 + \int_{t_2}^{t_3} \vec{F}_{ext}\, dt$

Not useful here: O_x & O_y are unknown

FBD during impact
from position ② to position ③



But $\vec{H}_o$ is conserved during impact:

$$(\vec{H}_o)_3 = (\vec{H}_o)_2 + \int_{t_2}^{t_3} \vec{M}_o^{\,0}\, dt \qquad \text{(for plug \& sphere)}$$

Note:

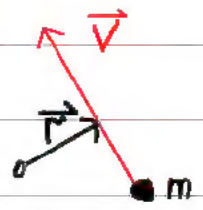
- For a point mass m going in a circle of radius r about o

$$|\vec{H}_o| = r\,mv = m r^2 \dot{\theta} \quad , v = r\dot{\theta}$$

Also:

- For mass m going past o at a perpendicular distance r:

$$\vec{H}_o = \vec{r} \times m\vec{v} \rightarrow |\vec{H}_o| = r\,mv(\sin\theta)$$



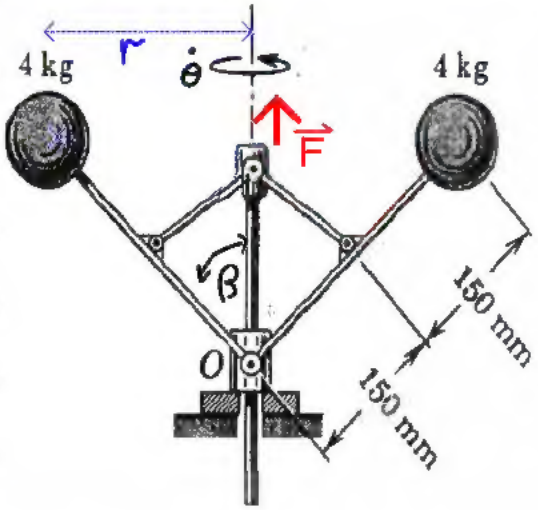
$$\rightarrow \left[(4\text{kg} + 2\text{kg})(0.5 \text{ meters})^2 + (6\text{kg})(0.3\text{meters})^2 \right] \dot{\theta}_3 =$$

$$(4\text{kg})(0.5\text{ meters})^2 (+2\text{s}^{-1}) + (6\text{kg})(0.3\text{ meters})^2 (+2\text{s}^{-1}) - (2\text{kg})(3.43\text{ m/s})(0.5\text{ m})$$

$$\rightarrow \boxed{\dot{\theta}_3 = -0.172\ \text{s}^{-1}}$$

Eg:

system is rotating freely at $\dot{\theta}_1 = 90$ rev/minute, $\beta = 60°$.



Force $\vec{F}$ is increased until $\beta = 30°$. Determine new rotational speed $\dot{\theta}_2$ and work done by $\vec{F}$.

eqn (1): $\boxed{(\vec{H}_o)_2 = (\vec{H}_o)_1 + \cancelto{0}{\int_{t_1}^{t} \vec{M}_o \, dt}}$, where $|\vec{H}_o| = r m \cancelto{r\dot{\theta}}{v}$

$$\underbrace{(2)(4\text{kg})}_{m}\underbrace{(\dot{\theta}_2)}_{\dot{\theta}_2}\underbrace{(0.3 \sin 30)^2}_{r_2^2} = \underbrace{(2)(4\text{ kg})}_{m}\underbrace{(90\text{s}^{-1})(\tfrac{2\pi}{60})}_{\dot{\theta}_1}\underbrace{(0.3 \sin 60)^2}_{r_1^2}$$

$$\rightarrow \dot{\theta}_2 = 28.3 \text{ s}^{-1}$$

eqn (2): $\boxed{T_2 = T_1 + U_{1\to 2}}$

$$2 \text{ spheres}\left[\tfrac{1}{2} m (r\dot{\theta})^2\right]_2 = 2 \text{ spheres}\left[\tfrac{1}{2} m (r\dot{\theta})^2\right]_1 +$$

$$+ \text{ work done by } \vec{F} + 2 \text{ spheres}\left[-mg\Delta h\right]$$

$$\rightarrow 2\left[(\tfrac{1}{2})(4\text{kg})(\tfrac{0.3}{2} \times 28.3)^2\right] = 2\left[(\tfrac{1}{2})(4\text{kg})\left((0.3)(\sin 60°)(90)(\tfrac{2\pi}{60})\right)^2\right]$$

$$+ \text{ work done by } \vec{F} + 2\left[-(4\text{kg})(9.81)\underbrace{(0.3\cos 30° - 0.3\cos 60°)}_{\Delta h > 0}\right]$$

$\rightarrow$ $\boxed{\text{work done by } \vec{F} \text{ is } 56.6 \text{ Newton·meters}}$

Summary

For a system of N particles,

- $$\vec{H}_o = \sum_{i=1}^{N} (\vec{H}_o)_i = (\vec{r}_i \times m_i \vec{v}_i)$$
- $$\rightarrow (\vec{H}_o)_2 = (\vec{H}_o)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (\vec{M}_o)_i \, dt$$

Also,

- $$\vec{H}_G = \sum_{i=1}^{N} (\vec{\rho}_i \times m_i \vec{v}_i)$$
- $$(\vec{H}_G)_2 = (\vec{H}_G)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (\vec{M}_G)_i \, dt$$
- $$(\vec{H}_o)_2 = (\vec{H}_o)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (\vec{M}_o)_i \, dt$$

Next Lecture

Part II ! 2D (planar) Bodies

- Read 5/1 – 5/3